Boolean algebra
Combinational-circuit analysis

Lecture 2

Boolean algebra

- a.k.a. "switching algebra"
  - deals with boolean values -- 0, 1
- Positive-logic convention
  - analog voltages LOW, HIGH --> 0, 1
- Negative logic -- seldom used
- Signal values denoted by variables (X, Y, FRED, etc.)

Boolean operators

- Complement: X' (opposite of X)
- AND: $X \cdot Y$
- OR: $X + Y$

binary operators, described functionally by truth table.

X	Y	X AND Y
0	0	0
0	1	0
1	0	0
1	1	1

X	Y	X OR Y
0	0	0
0	1	1
1	0	1
1	1	1

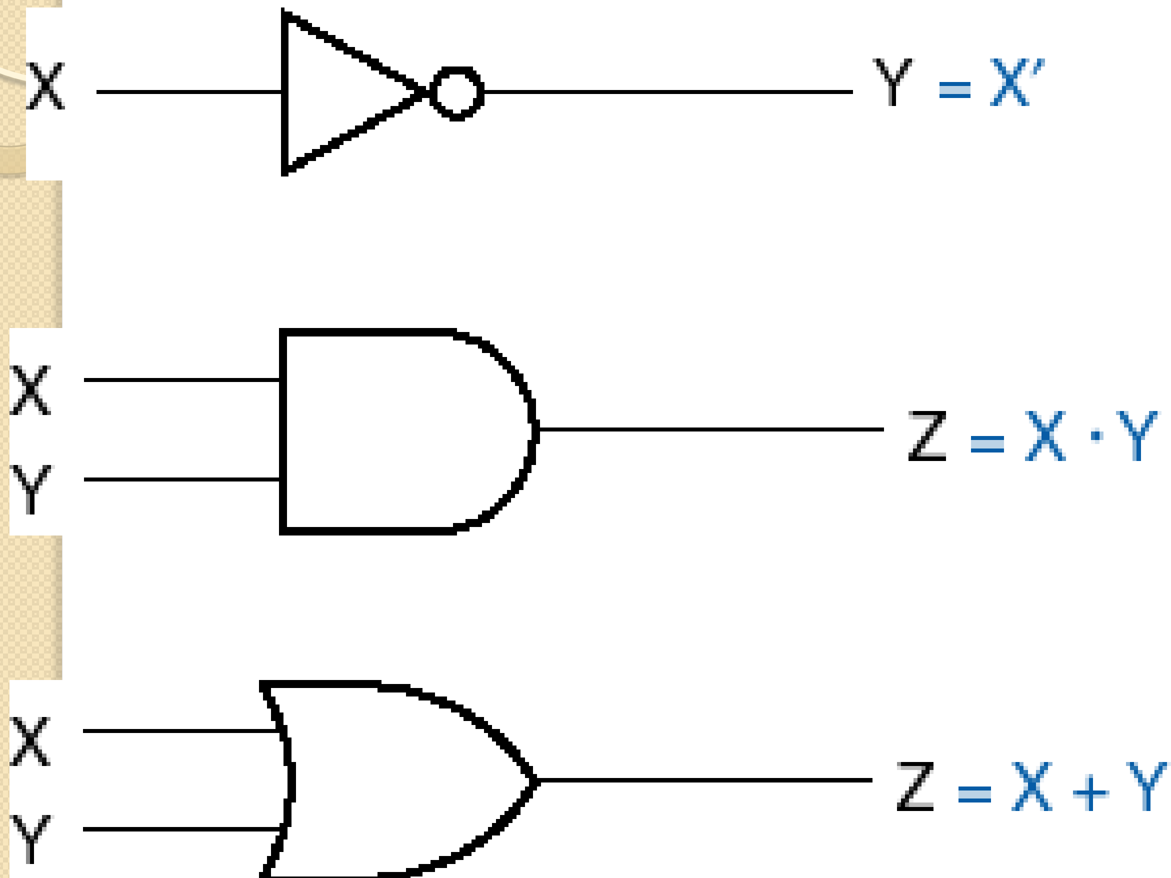
X	NOT X
0	1
1	0

- Axiomatic definition: A1-A5, A1'-A5'

More definitions

- Literal: a variable or its complement
  - X, X', FRED', CS_L
- Expression: literals combined by AND, OR, parentheses, complementation
  - X+Y
  - $P \cdot Q \cdot R$
  - $A + B \cdot C$
  - $((FRED \cdot Z') + CS\_L \cdot A \cdot B' \cdot C + Q5) \cdot RESET'$
- Equation: Variable = expression
  - $P = ((FRED \cdot Z') + CS\_L \cdot A \cdot B' \cdot C + Q5) \cdot RESET'$

Logic symbols



Theorems

(T1)	$X + 0 = X$	(T1′)	$X \cdot 1 = X$	(Identities)
(T2)	$X + 1 = 1$	(T2′)	$X \cdot 0 = 0$	(Null elements)
(T3)	$X + X = X$	(T3′)	$X \cdot X = X$	(Idempotency)
(T4)	$(X')' = X$			(Involution)
(T5)	$X + X' = 1$	(T5′)	$X \cdot X' = 0$	(Complements)

- Proofs by perfect induction

More Theorems

(T6)	$X + Y = Y + X$	(T6′)	$X \cdot Y = Y \cdot X$	(Commutativity)
(T7)	$(X + Y) + Z = X + (Y + Z)$	(T7′)	$(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$	(Associativity)
(T8)	$X \cdot Y + X \cdot Z = X \cdot (Y + Z)$	(T8′)	$(X + Y) \cdot (X + Z) = X + Y \cdot Z$	(Distributivity)
(T9)	$X + X \cdot Y = X$	(T9′)	$X \cdot (X + Y) = X$	(Covering)
(T10)	$X \cdot Y + X \cdot Y' = X$	(T10′)	$(X + Y) \cdot (X + Y') = X$	(Combining)
(T11)	$X \cdot Y + X' \cdot Z + Y \cdot Z = X \cdot Y + X' \cdot Z$			(Consensus)
(T11′)	$(X + Y) \cdot (X' + Z) \cdot (Y + Z) = (X + Y) \cdot (X' + Z)$			

- N.B. T8′, T10, T11

Duality

- Swap 0 & 1, AND & OR
  - Result: Theorems still true
- Why?
  - Each axiom (A1-A5) has a dual (A1′-A5′)
- Counterexample:

$X + (X \cdot Y) = X$ (T9)

$X \cdot (X + Y) = X$ (dual)

$X + (X . Y) = X$ (T3′)

$X + (X \cdot Y) = X$ (T9)
$X \cdot (X + Y) = X$ (dual)
$(X \cdot X) + (X \cdot Y) = X$ (T8)
$X + (X \cdot Y) = X$ (T3′)
parentheses,
operator precedence!

N-variable Theorems

(T12) $X + X + \cdots + X = X$ (Generalized idempotency)

(T12′) $X \cdot X \cdot \ \cdots \ \cdot X = X$

(T13) $(X_1 \cdot X_2 \cdot \ \cdots \ \cdot X_n)' = X_1' + X_2' + \cdots + X_n'$ (DeMorgan's theorems)

(T13′) $(X_1 + X_2 + \cdots + X_n)' = X_1' \cdot X_2' \cdot \ \cdots \ \cdot X_n'$

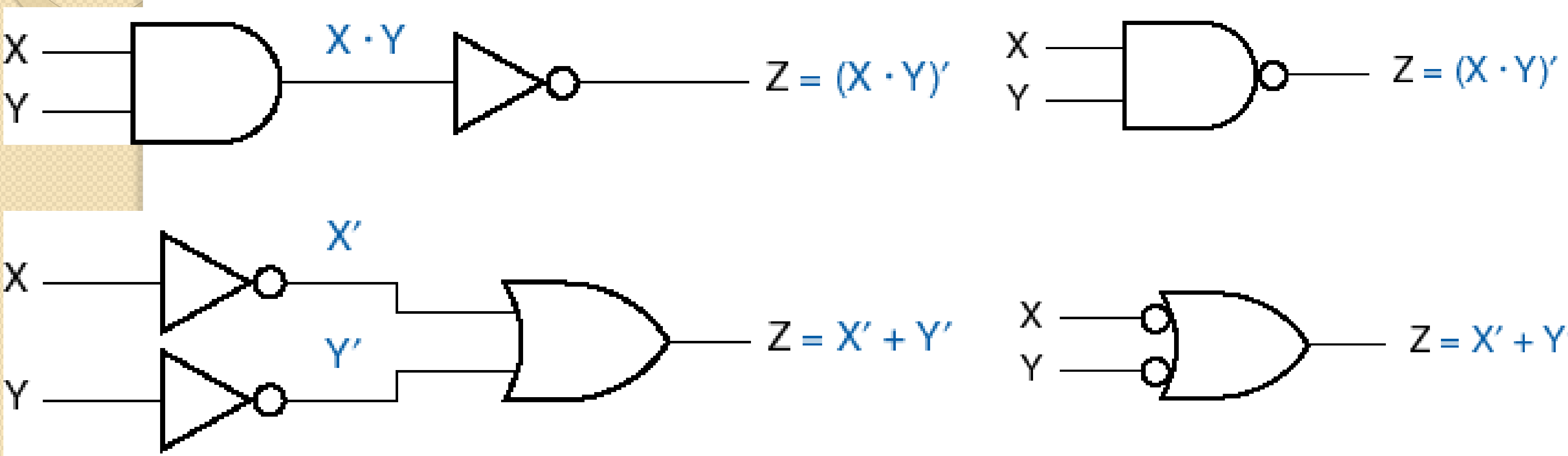
(T14) $[F(X_1, X_2, \ldots, X_n, +, \cdot)]' = F(X_1', X_2', \ldots, X_n', \cdot, +)$ (Generalized DeMorgan's theorem)

(T15) $F(X_1, X_2, \ldots, X_n) = X_1 \cdot F(1, X_2, \ldots, X_n) + X_1' \cdot F(0, X_2, \ldots, X_n)$ (Shannon's expansion theorems)

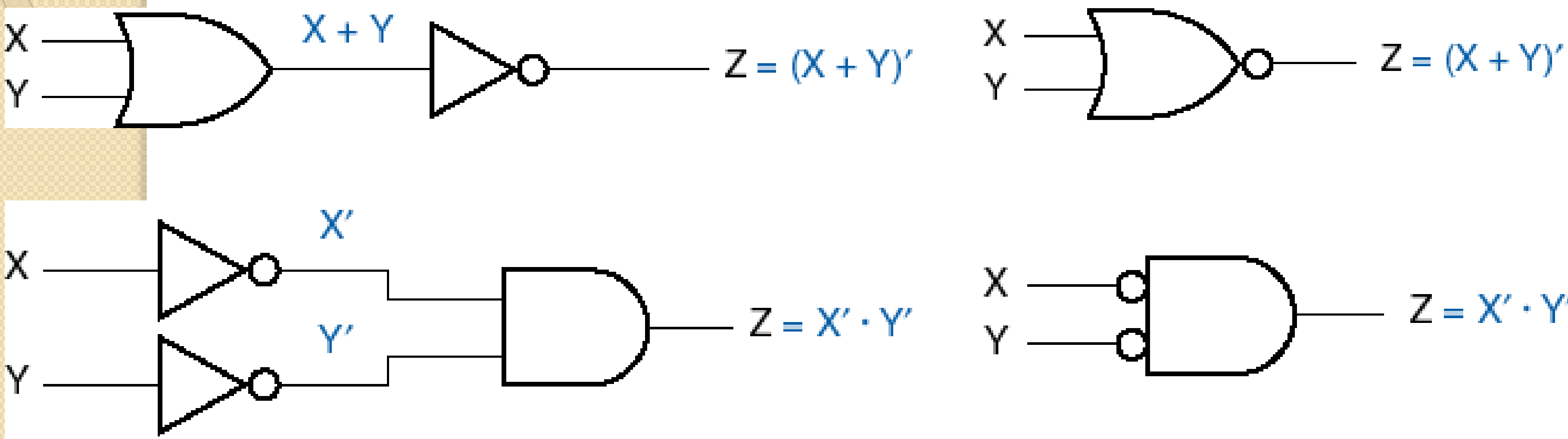
(T15′) $F(X_1, X_2, \ldots, X_n) = [X_1 + F(0, X_2, \ldots, X_n)] \cdot [X_1' + F(1, X_2, \ldots, X_n)]$

- Prove using finite induction
- Most important: DeMorgan theorems

DeMorgan Symbol Equivalence

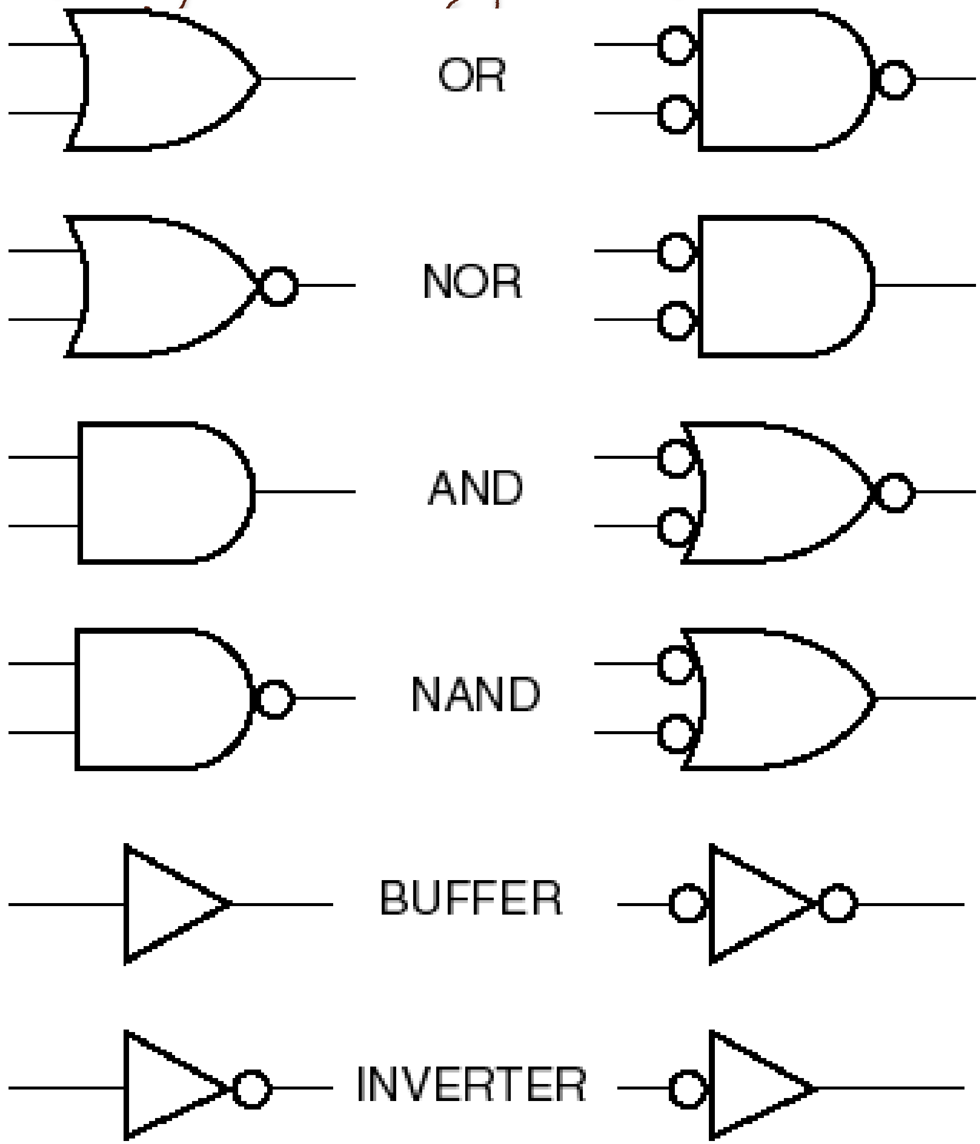


Likewise for OR



- (be sure to check errata!)

DeMorgan Symbols



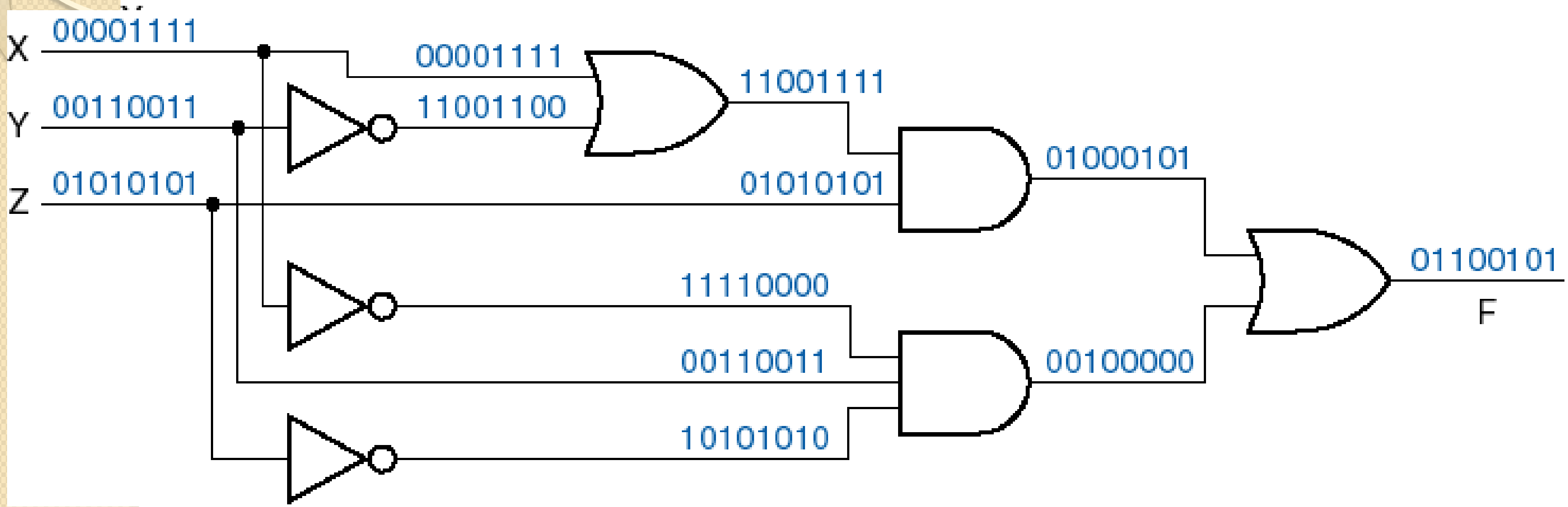
Even more definitions (Sec. 4.1.6)

- Product term
- Sum-of-products expression
- Sum term
- Product-of-sums expression
- Normal term
- Minterm (n variables)
- Maxterm (n variables)

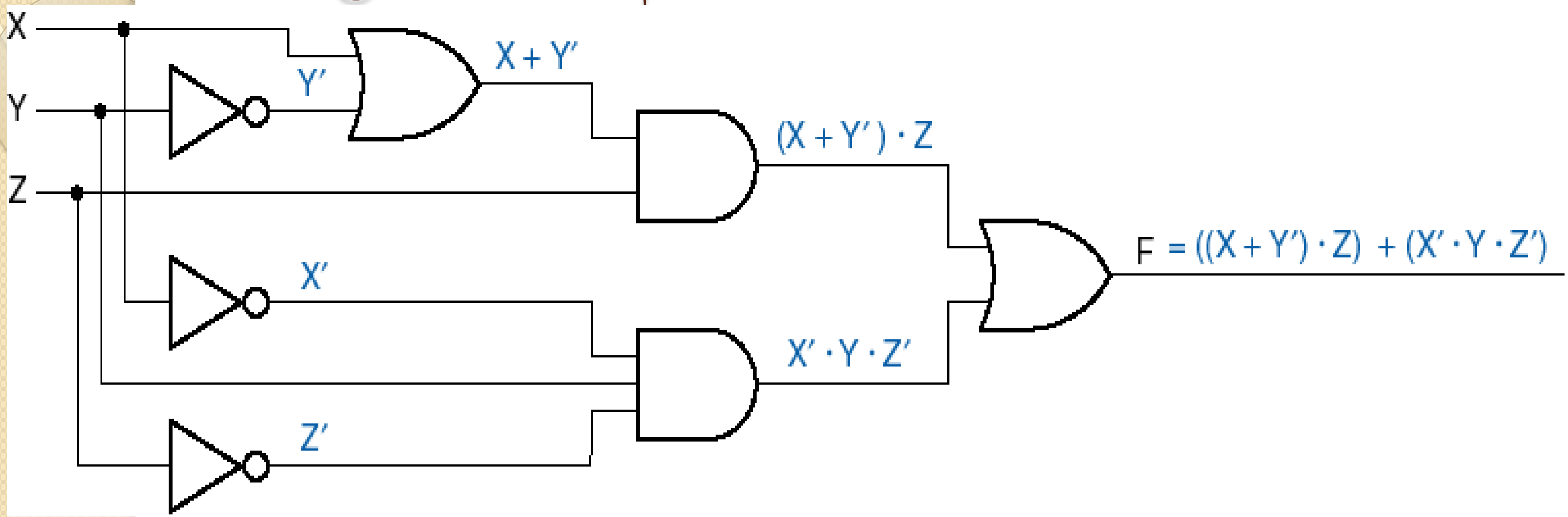
Truth table vs. minterms & maxterms

Row	X	Y	Z	F	*Minterm*	*Maxterm*
0	0	0	0	F(0,0,0)	$X' \cdot Y' \cdot Z'$	$X + Y + Z$
1	0	0	1	F(0,0,1)	$X' \cdot Y' \cdot Z$	$X + Y + Z'$
2	0	1	0	F(0,1,0)	$X' \cdot Y \cdot Z'$	$X + Y' + Z$
3	0	1	1	F(0,1,1)	$X' \cdot Y \cdot Z$	$X + Y' + Z'$
4	1	0	0	F(1,0,0)	$X \cdot Y' \cdot Z'$	$X' + Y + Z$
5	1	0	1	F(1,0,1)	$X \cdot Y' \cdot Z$	$X' + Y + Z'$
6	1	1	0	F(1,1,0)	$X \cdot Y \cdot Z'$	$X' + Y' + Z$
7	1	1	1	F(1,1,1)	$X \cdot Y \cdot Z$	$X' + Y' + Z'$

Combinational analysis



Signal expressions

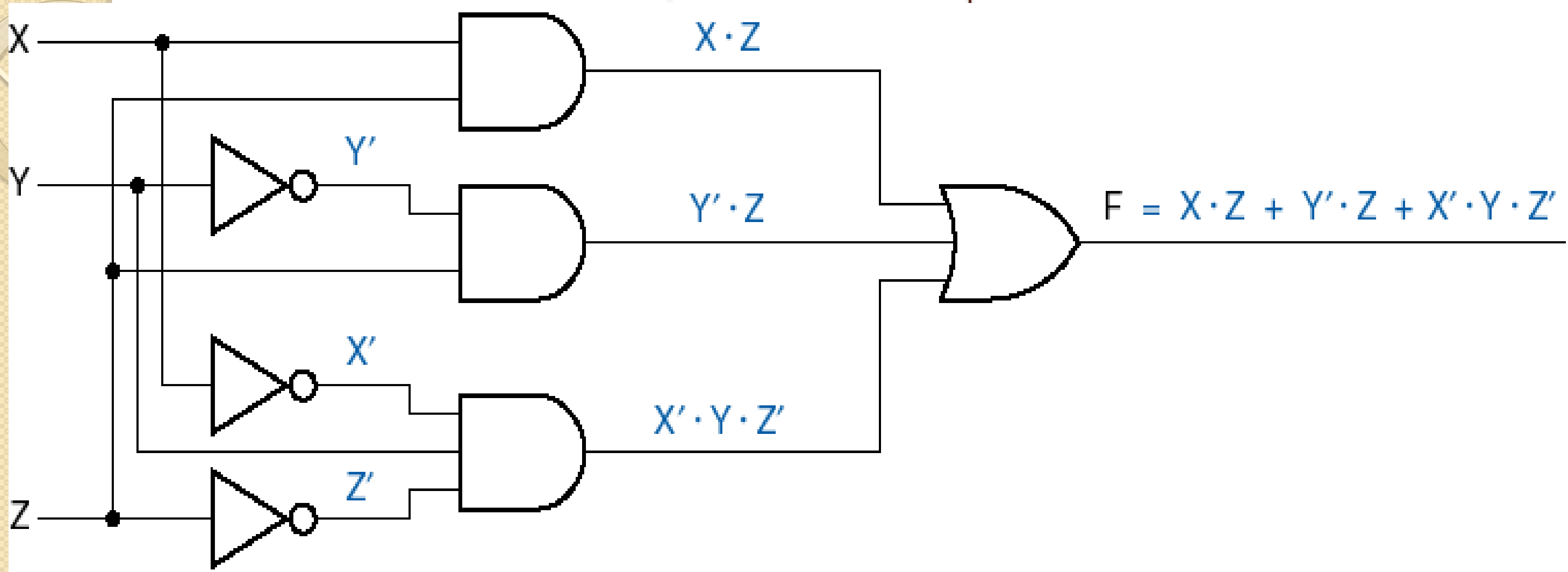


- Multiply out:

$$F = ((X + Y') \cdot Z) + (X' \cdot Y \cdot Z')$$

$$= (X \cdot Z) + (Y' \cdot Z) + (X' \cdot Y \cdot Z')$$

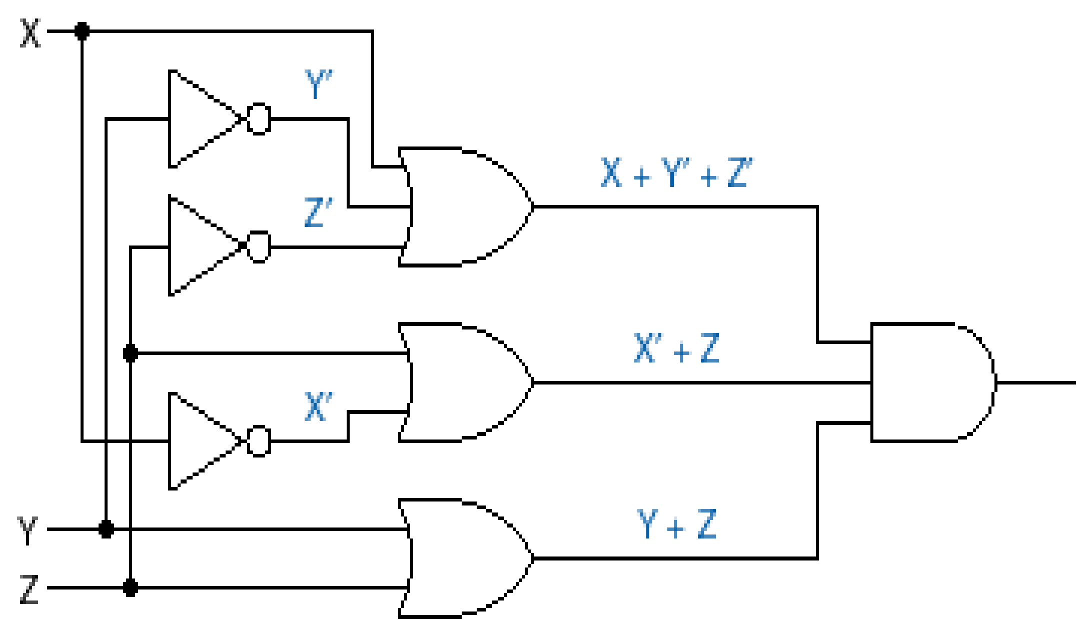
New circuit, same function



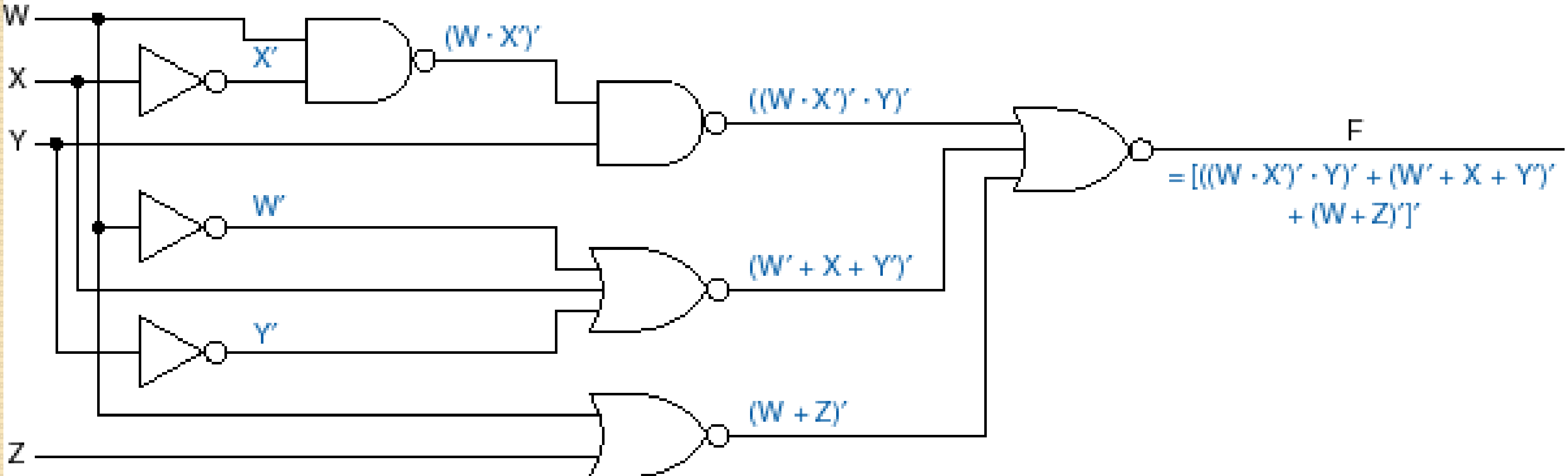
“Add out” logic function

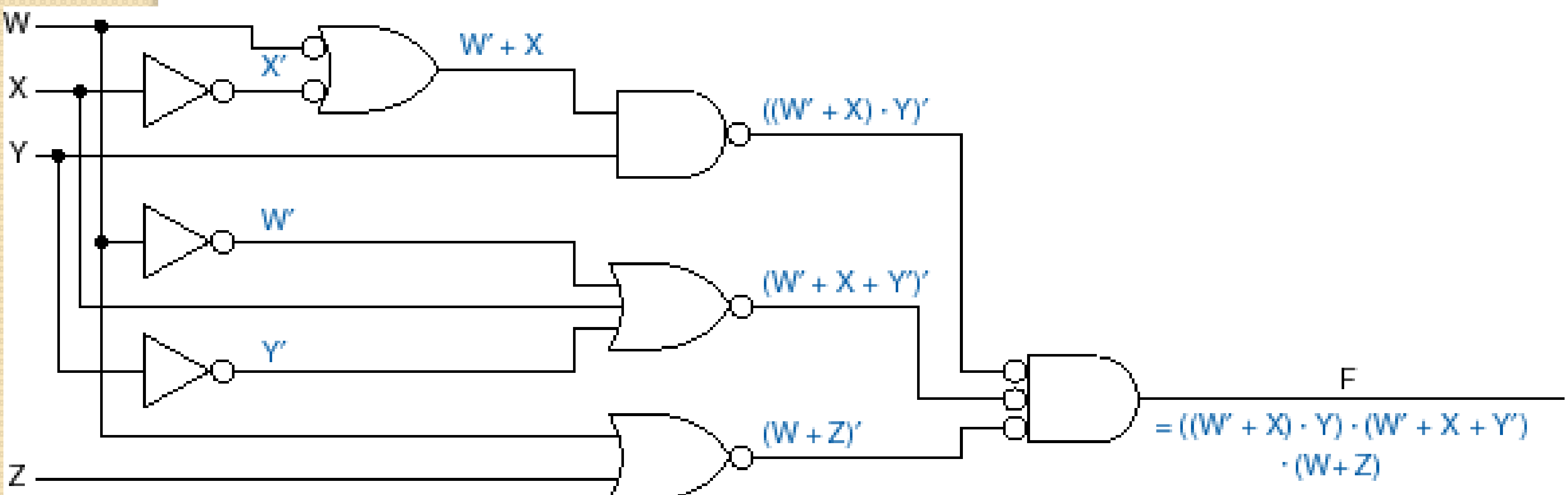
$$F = ((X + Y') \cdot Z) + (X' \cdot Y \cdot Z')$$
$$= (X + Y' + X') \cdot (X + Y' + Y) \cdot (X + Y' + Z') \cdot (Z + X') \cdot (Z + Y) \cdot (Z + Z')$$
$$= 1 \cdot 1 \cdot (X + Y' + Z') \cdot (X' + Z) \cdot (Y + Z) \cdot 1$$
$$= (X + Y' + Z') \cdot (X' + Z) \cdot (Y + Z)$$

- Circuit:

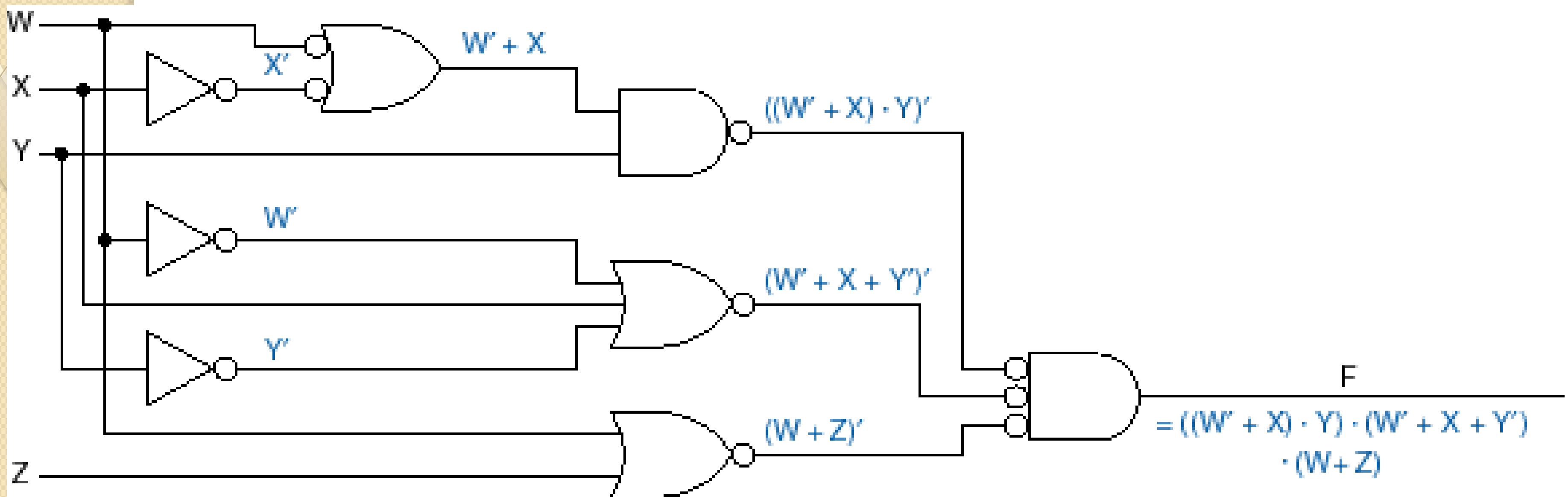


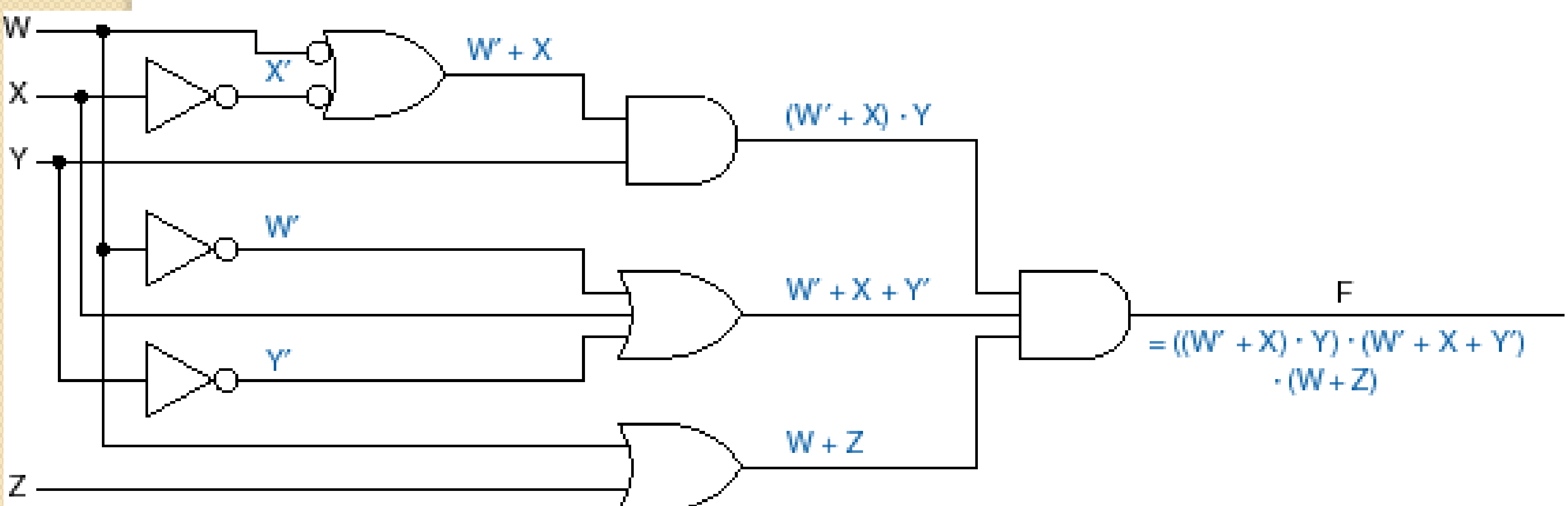
Shortcut: Symbol substitution





Different circuit, same function





Another example

$$G(W, X, Y, Z) = W \cdot X \cdot Y + Y \cdot Z$$