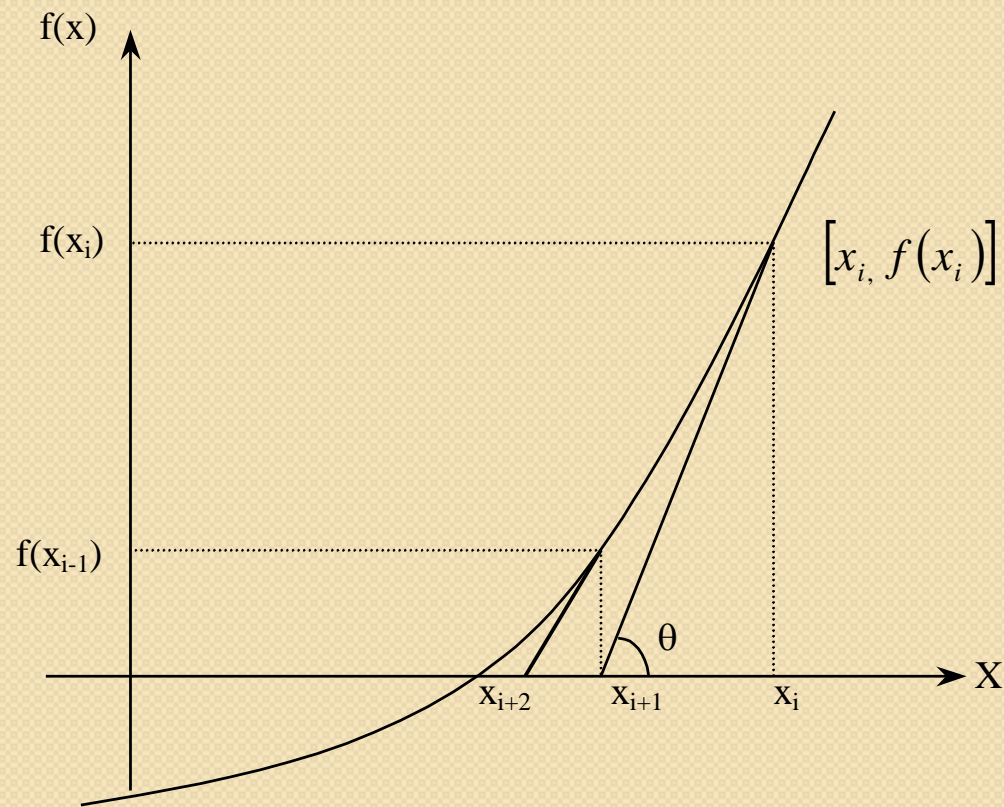


# Newton-Raphson Method



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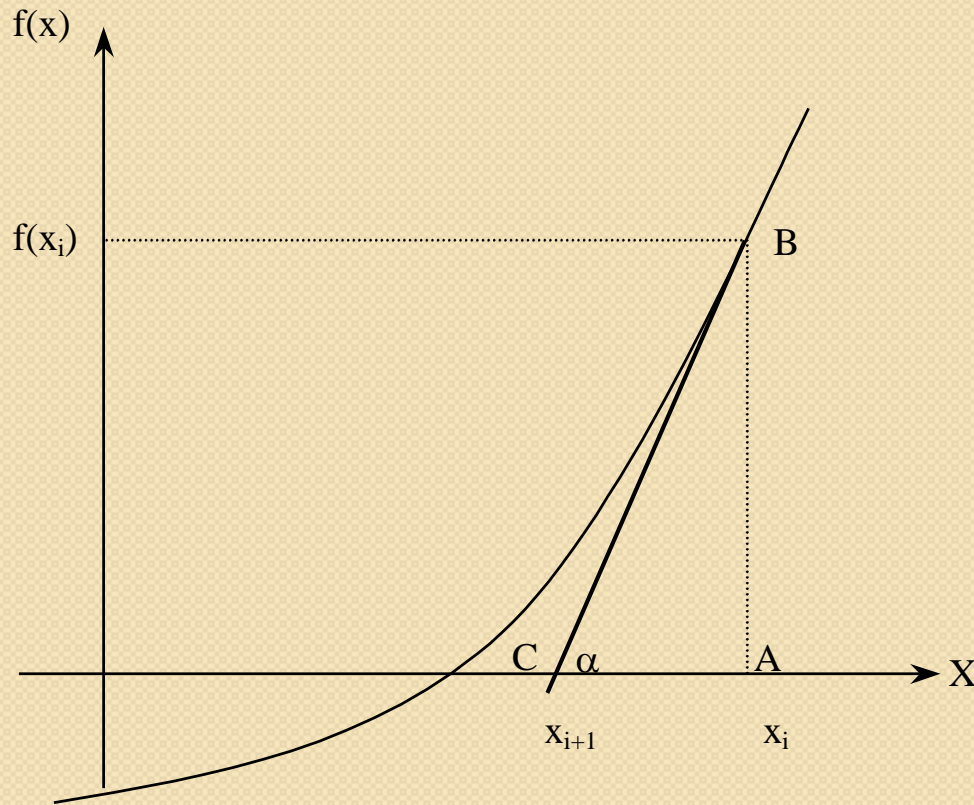
# Newton-Raphson Method



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 1 Geometrical illustration of the Newton-Raphson method.

# Derivation



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 2 Derivation of the Newton-Raphson method.

# Algorithm for Newton-Raphson Method

Step 1

Evaluate  $f'(x)$  symbolically.

## Step 2

use an initial guess of the root,  $x_i$  to estimate the new value of the root,  $x_{i+1}$ , as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

## Step 3

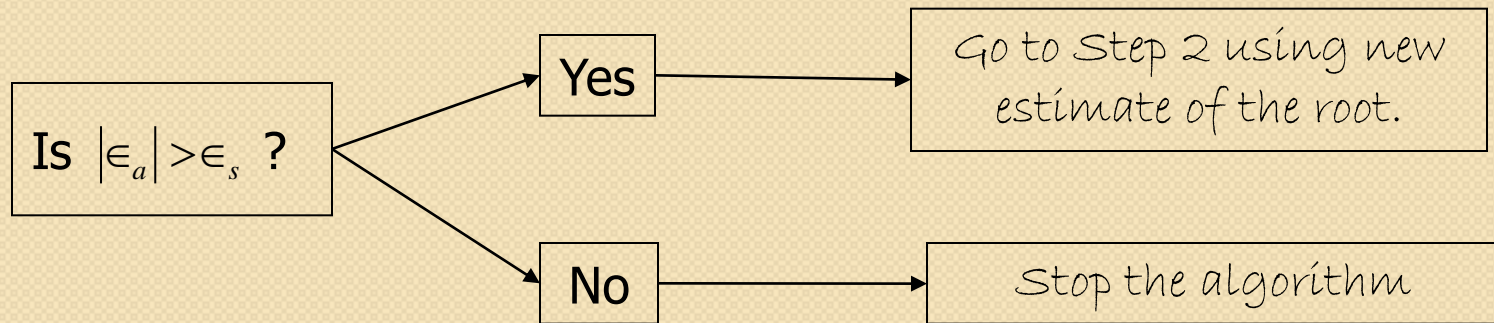
Find the absolute relative approximate error  $|\epsilon_a|$  as

$$|\epsilon_a| = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100$$



# Step 4

Compare the absolute relative approximate error with the pre-specified relative error tolerance  $\epsilon_s$ .



Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

# Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.

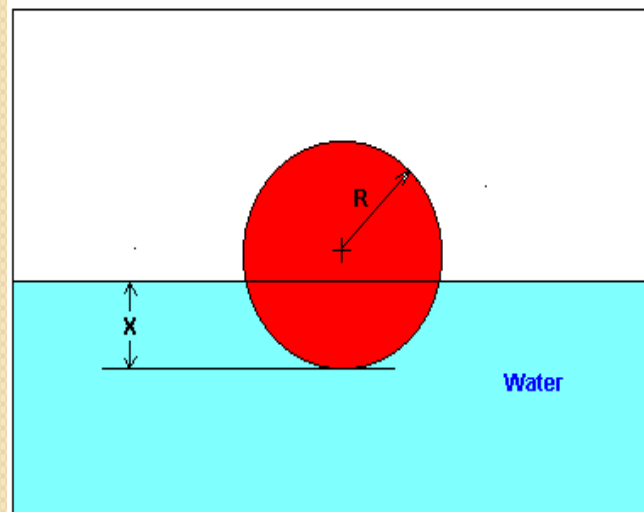


Figure 3 Floating ball problem.

# Example 1 Cont.

The equation that gives the depth  $x$  in meters to which the ball is submerged under water is given by

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

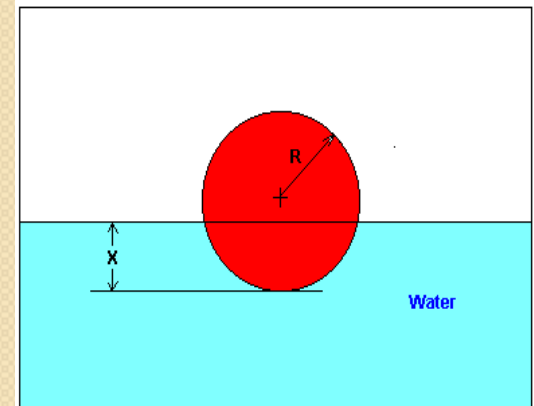


Figure 3 Floating ball problem.

Use the Newton's method of finding roots of equations to find

- the depth ' $x$ ' to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation.
- The absolute relative approximate error at the end of each iteration, and
- The number of significant digits at least correct at the end of each iteration.

# Example 1 Cont.

## Solution

To aid in the understanding of how this method works to find the root of an equation, the graph of  $f(x)$  is shown to the right,

where

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

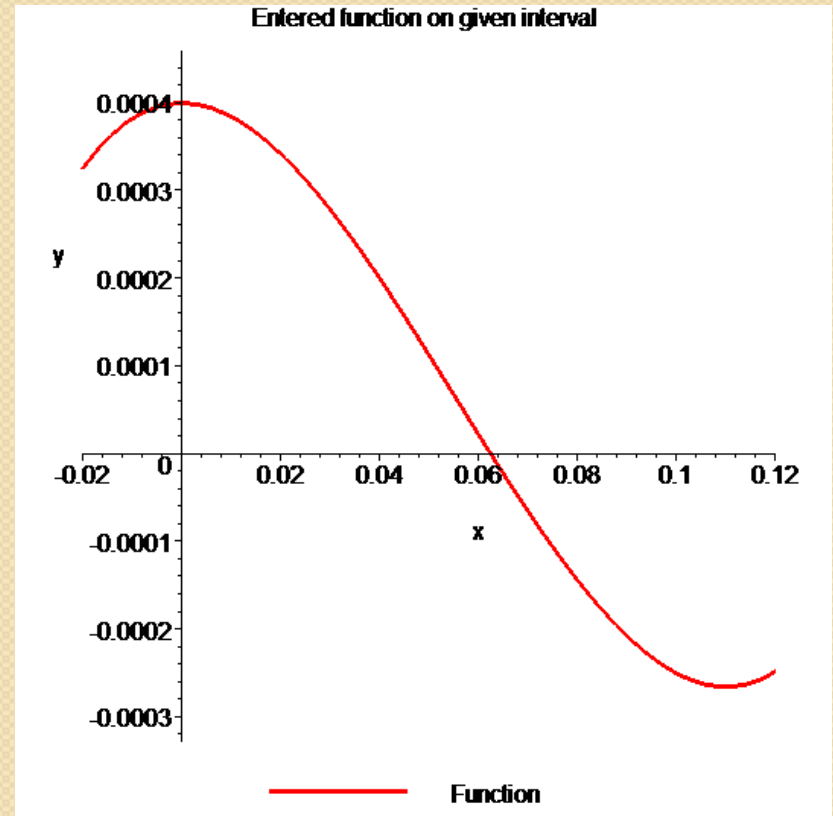


Figure 4 Graph of the function  $f(x)$

## Example 1 Cont.

Solve for  $f'(x)$

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

$$f'(x) = 3x^2 - 0.33x$$

Let us assume the initial guess of the root of  $f(x) = 0$  is  $x_0 = 0.05\text{m}$ . This is a reasonable guess (discuss why  $x = 0$  and  $x = 0.1\text{m}$  are not good choices) as the extreme values of the depth  $x$  would be 0 and the diameter (0.11 m) of the ball.

# Example 1 Cont.

Iteration 1

The estimate of the root is

$$\begin{aligned}x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\&= 0.05 - \frac{(0.05)^3 - 0.165(0.05)^2 + 3.993 \times 10^{-4}}{3(0.05)^2 - 0.33(0.05)} \\&= 0.05 - \frac{1.118 \times 10^{-4}}{-9 \times 10^{-3}} \\&= 0.05 - (-0.01242) \\&= 0.06242\end{aligned}$$

# Example 1 Cont.

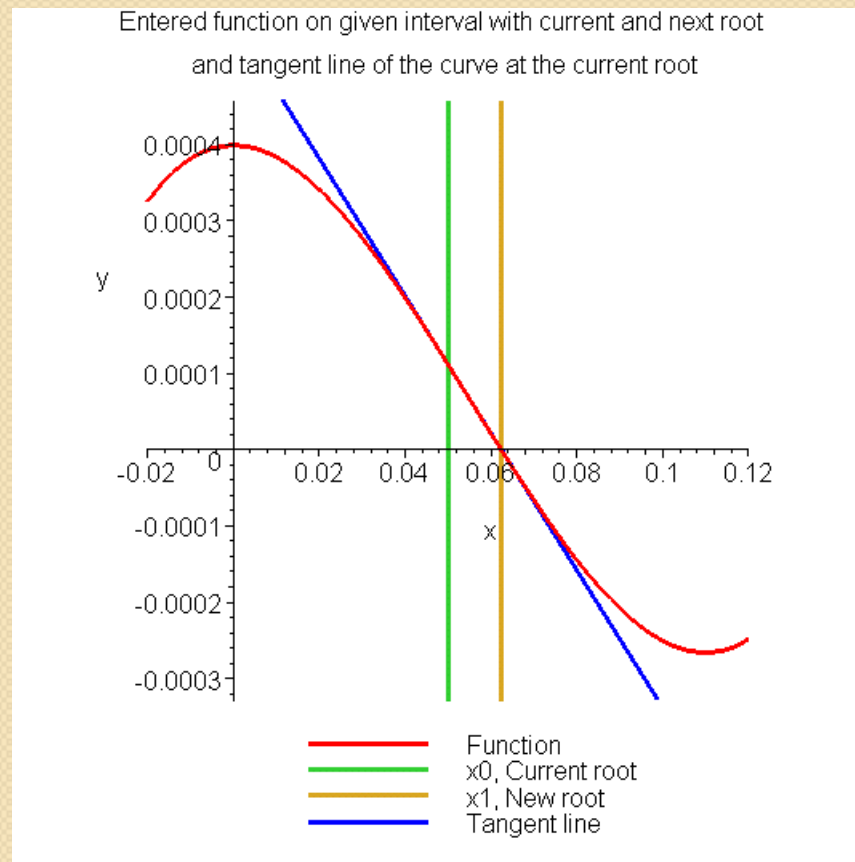


Figure 5 Estimate of the root for the first iteration.

## Example 1 Cont.

The absolute relative approximate error  $|\epsilon_a|$  at the end of iteration 1 is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{x_1 - x_0}{x_1} \right| \times 100 \\ &= \left| \frac{0.06242 - 0.05}{0.06242} \right| \times 100 \\ &= 19.90\% \end{aligned}$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for at least one significant digits to be correct in your result.



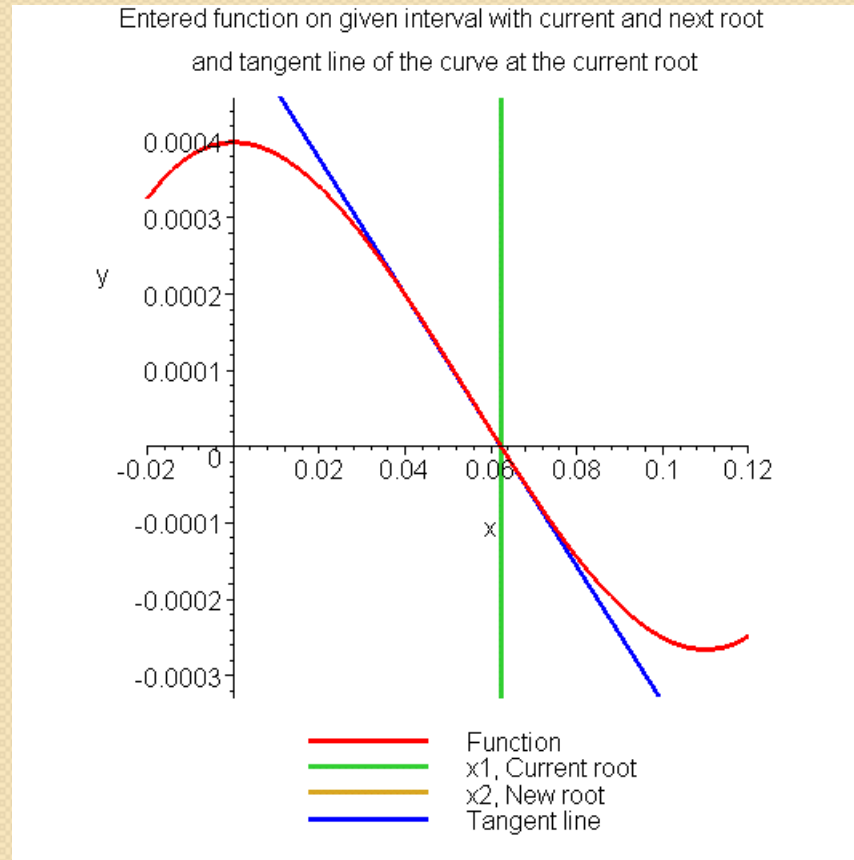
# Example 1 Cont.

Iteration 2

The estimate of the root is

$$\begin{aligned}x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\&= 0.06242 - \frac{(0.06242)^3 - 0.165(0.06242)^2 + 3.993 \times 10^{-4}}{3(0.06242)^2 - 0.33(0.06242)} \\&= 0.06242 - \frac{-3.97781 \times 10^{-7}}{-8.90973 \times 10^{-3}} \\&= 0.06242 - (4.4646 \times 10^{-5}) \\&= 0.06238\end{aligned}$$

# Example 1 Cont.



**Figure 6** Estimate of the root for the Iteration 2.

## Example 1 Cont.

The absolute relative approximate error  $|\epsilon_a|$  at the end of iteration 2 is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{x_2 - x_1}{x_2} \right| \times 100 \\ &= \left| \frac{0.06238 - 0.06242}{0.06238} \right| \times 100 \\ &= 0.0716\% \end{aligned}$$

The maximum value of  $m$  for which  $|\epsilon_a| \leq 0.5 \times 10^{2-m}$  is 2.844. Hence, the number of significant digits at least correct in the answer is 2.

# Example 1 Cont.

Iteration 3

The estimate of the root is

$$\begin{aligned}x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\&= 0.06238 - \frac{(0.06238)^3 - 0.165(0.06238)^2 + 3.993 \times 10^{-4}}{3(0.06238)^2 - 0.33(0.06238)} \\&= 0.06238 - \frac{4.44 \times 10^{-11}}{-8.91171 \times 10^{-3}} \\&= 0.06238 - (-4.9822 \times 10^{-9}) \\&= 0.06238\end{aligned}$$

# Example 1 Cont.

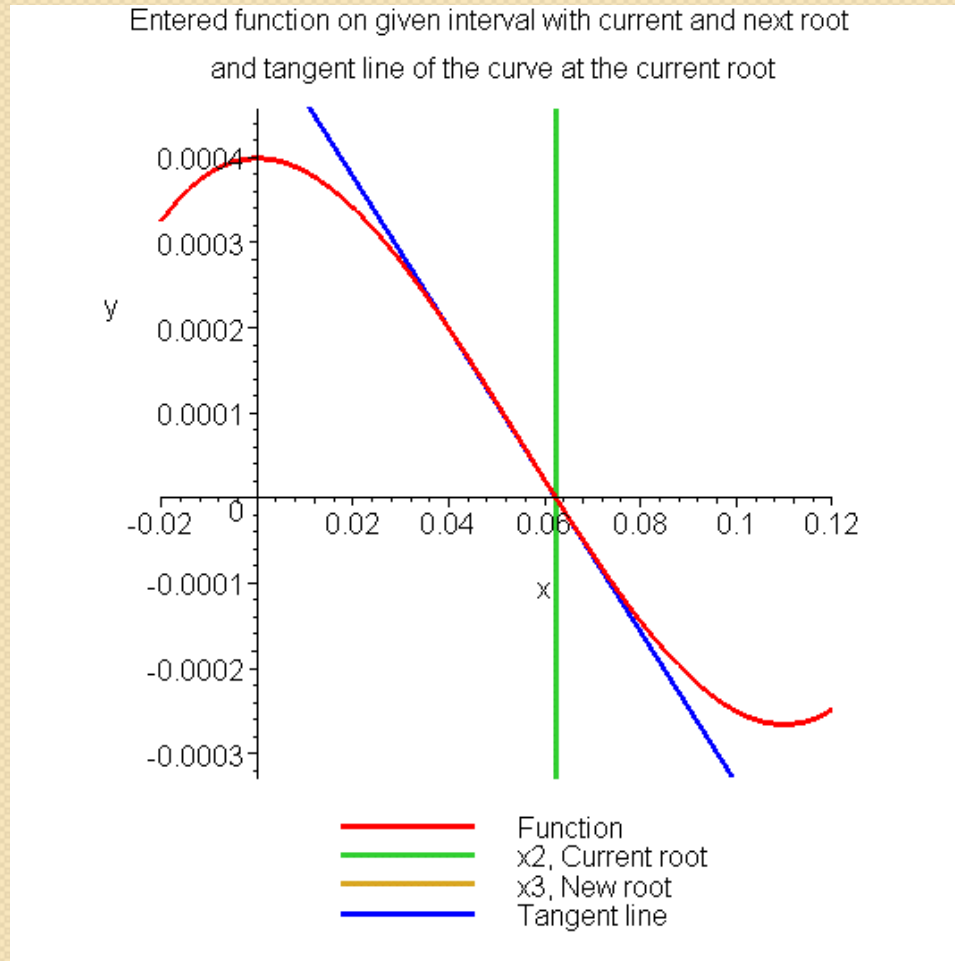


Figure 7 Estimate of the root for the iteration 3.

## Example 1 Cont.

The absolute relative approximate error  $|\epsilon_a|$  at the end of iteration 3 is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{x_2 - x_1}{x_2} \right| \times 100 \\ &= \left| \frac{0.06238 - 0.06238}{0.06238} \right| \times 100 \\ &= 0\% \end{aligned}$$

The number of significant digits at least correct is 4, as only 4 significant digits are carried through all the calculations.

# Advantages and Drawbacks of Newton Raphson Method

# Advantages

- Converges fast (quadratic convergence), if it converges.
- Requires only one guess



# Drawbacks

## 1. Divergence at inflection points

Selection of the initial guess or an iteration value of the root that is close to the inflection point of the function  $f(x)$  start diverging away from the root in the Newton-Raphson method.

For example, to find the root of the equation  $f(x) = (x-1)^3 + 0.512 = 0$

The Newton-Raphson method reduces to 
$$x_{i+1} = x_i - \frac{(x_i^3 - 1)^3 + 0.512}{3(x_i - 1)^2}$$

Table 1 shows the iterated values of the root of the equation.

The root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of  $x = 1$

Eventually after 12 more iterations the root converges to the exact value of  $x = 0.2$ .

# Drawbacks - Inflection Points

Table 1 Divergence near inflection point.

Iteration Number	$x_j$
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	-30.119
7	-19.746
18	0.2000

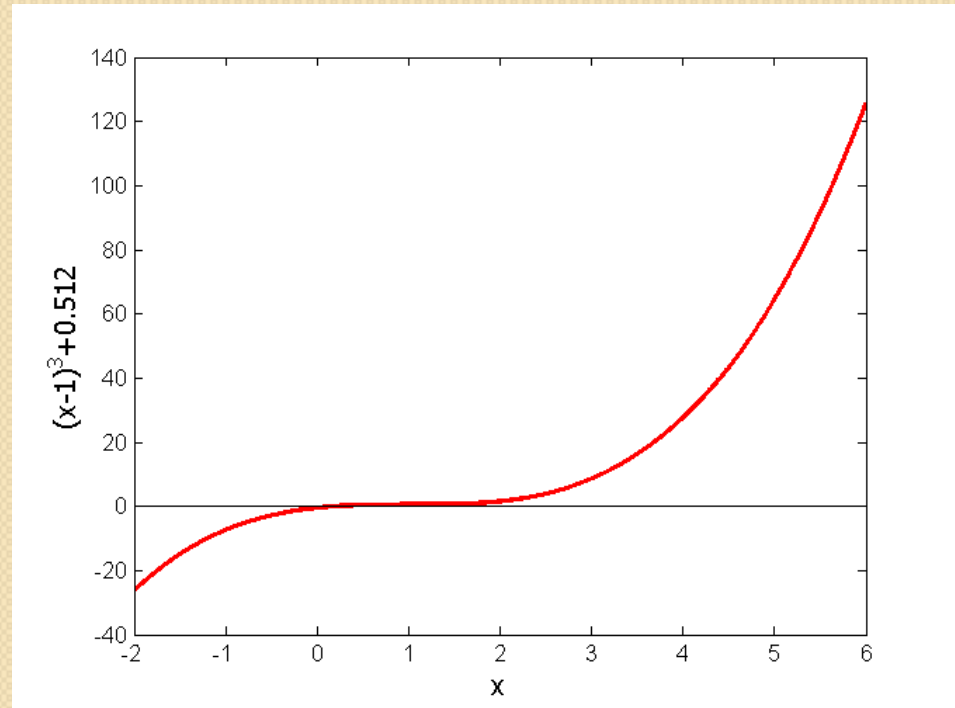


Figure 8 Divergence at inflection point for

$$f(x) = (x-1)^3 + 0.512 = 0$$

# Drawbacks - Division by Zero

## 2. Division by zero

For the equation

$$f(x) = x^3 - 0.03x^2 + 2.4 \times 10^{-6} = 0$$

the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4 \times 10^{-6}}{3x_i^2 - 0.06x_i}$$

For  $x_0 = 0$  or  $x_0 = 0.02$ , the denominator will equal zero.

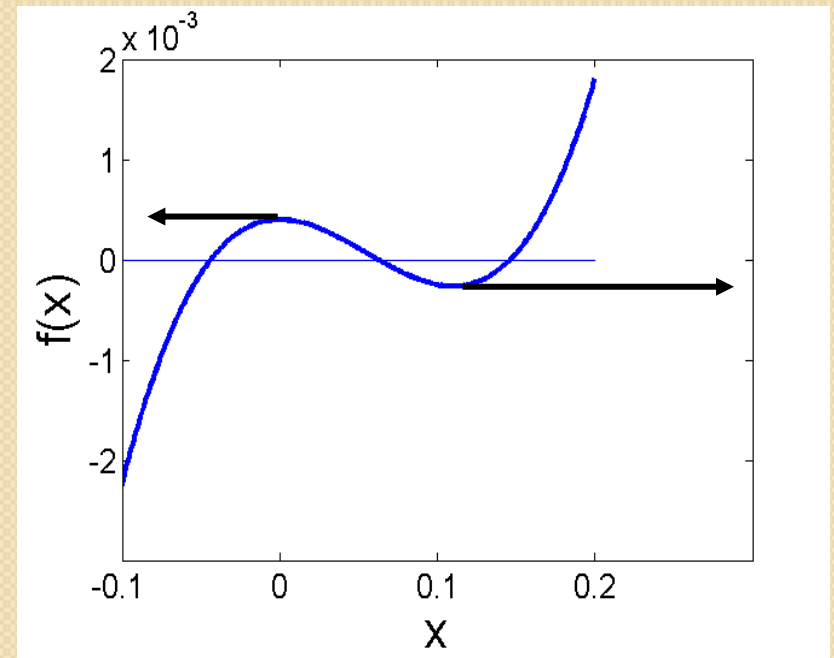


Figure 9 Pitfall of division by zero or near a zero number

# Drawbacks – Oscillations near local maximum and minimum

## 3. Oscillations near local maximum and minimum

Results obtained from the Newton-Raphson method may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum.

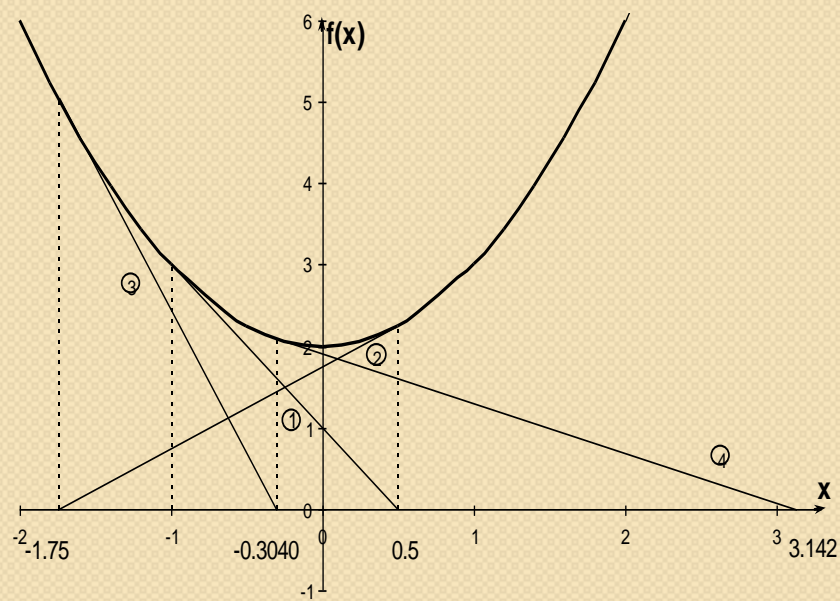
Eventually, it may lead to division by a number close to zero and may diverge.

For example for  $f(x) = x^2 + 2 = 0$  the equation has no real roots.

# Drawbacks – Oscillations near local maximum and minimum

**Table 3** Oscillations near local maxima and minima in Newton-Raphson method.

Iteration Number	$x_i$	$f(x_i)$	$ \epsilon_a \%$
0	-1.0000	3.00	
1	0.5	2.25	300.00
2	-1.75	5.063	128.571
3	-0.30357	2.092	476.47
4	3.1423	11.874	109.66
5	1.2529	3.570	150.80
6	-0.17166	2.029	829.88
7	5.7395	34.942	102.99
8	2.6955	9.266	112.93
9	0.97678	2.954	175.96



**Figure 10** Oscillations around local minima for  $f(x) = x^2 + 2$

# Drawbacks - Root Jumping

## 4. Root Jumping

In some cases where the function  $f(x)$  is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

For example

$$f(x) = \sin x = 0$$

Choose

$$x_0 = 2.4\pi = 7.539822$$

It will converge to  $x = 0$

instead of  $x = 2\pi = 6.2831853$

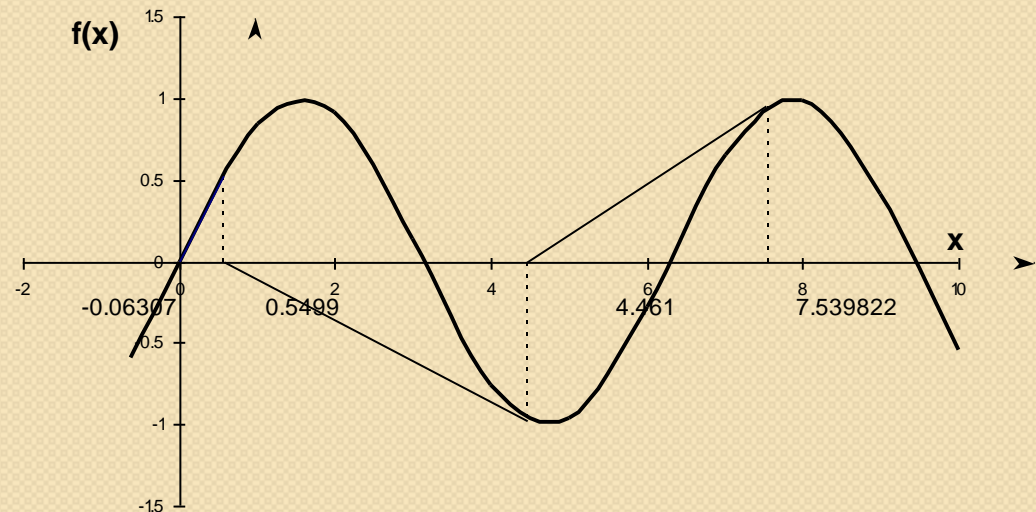


Figure 11 Root jumping from intended location of root for  $f(x) = \sin x = 0$



END of Presentation