Sources of Error

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Two sources of numerical error

1) Discretization Errors (Approximation errors)
2) Round-off error (Truncation errors)

Round-off Error

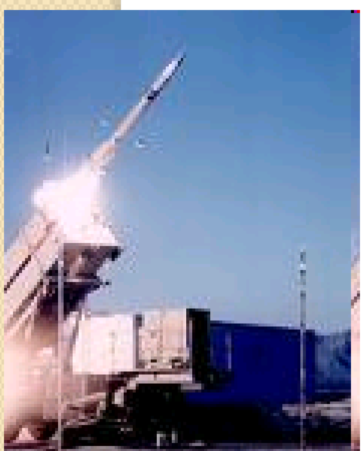
Round off Error

- Caused by representing a number approximately

$$\frac{1}{3} \cong 0.333333$$

$$\sqrt{2} \cong 1.4142...$$

Problem with Patriot missile



- Clock cycle of 1/10 seconds was represented in 24-bit fixed point register created an error of 9.5×10^{-8} seconds.
- The battery was on for 100 consecutive hours, thus causing an inaccuracy of

$$= 9.5\times10^{-8}\frac{\text{s}}{0.1\text{s}}\times100\text{hr}\times\frac{3600\text{s}}{1\text{hr}}$$
$$= 0.342s$$

Problem (cont.)

- The shift calculated in the ranging system of the missile was 687 meters.
- The target was considered to be out of range at a distance greater than 137 meters.

Effect of Carrying Significant Digits in Calculations

Discretization Error

Discretization error

- Error caused by approximating a mathematical procedure.

Example of Truncation Error

Taking only a few terms of a Maclaurin series to approximate e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots\dots\dots\dots$$

If only 3 terms are used,

$$Truncation\ Error = e^x - \left(1 + x + \frac{x^2}{2!}\right)$$

Another Example of Discretization Error

Using a finite Δx to approximate $f'(x)$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

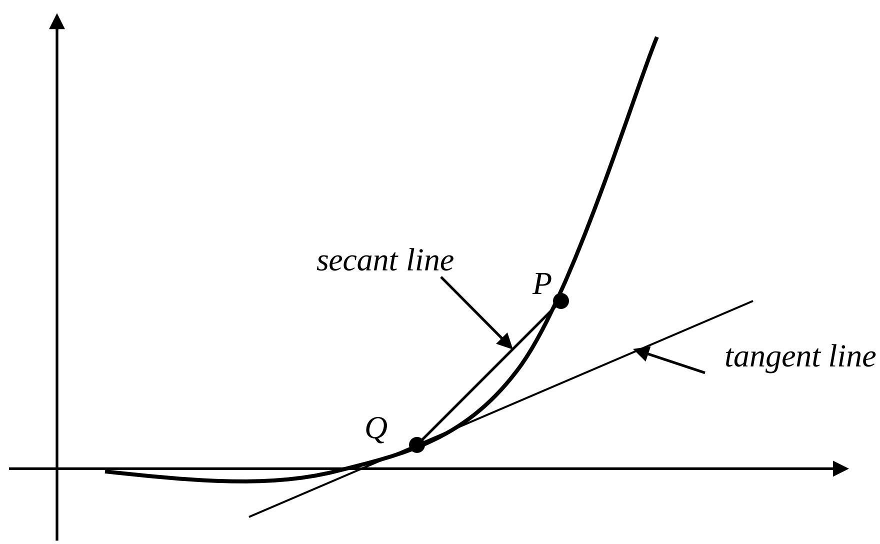
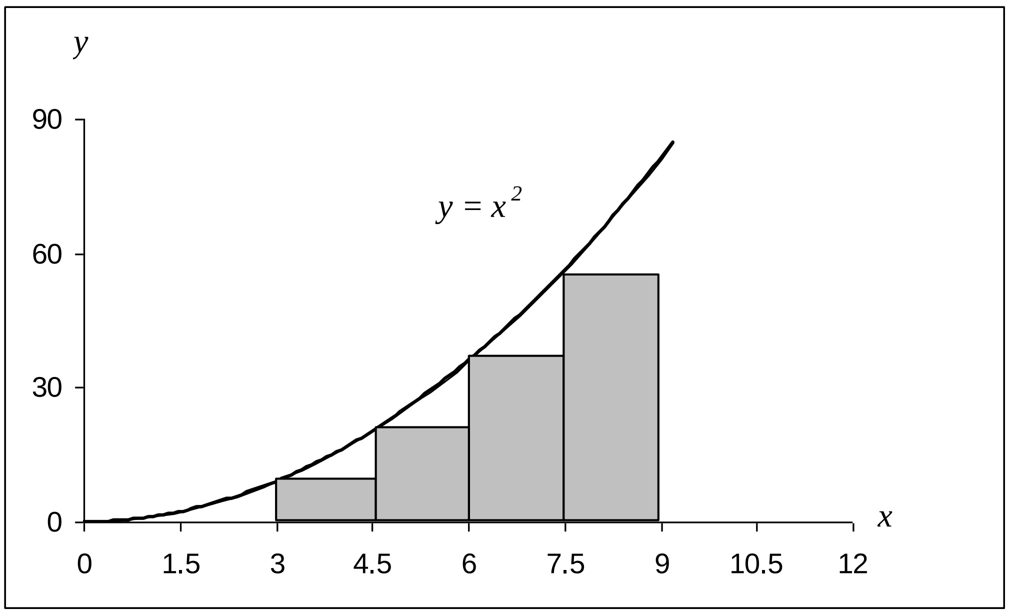


Figure 1. Approximate derivative using finite Δx

Another Example of Discretization Error

Using finite rectangles to approximate an integral.



Example 1 —Maclaurin series

Calculate the value of $e^{1.2}$ with an absolute relative approximate error of less than 1%.

$$e^{1.2} = 1 + 1.2 + \frac{1.2^2}{2!} + \frac{1.2^3}{3!} + \ldots\ldots\ldots\ \ldots\ldots\ldots$$

n	$e^{1.2}$	E_a	$\lvert \in_a \rvert \%$
1	1	—	—
2	2.2	1.2	54.545
3	2.92	0.72	24.658
4	3.208	0.288	8.9776
5	3.2944	0.0864	2.6226
6	3.3151	0.020736	0.62550

6 terms are required. How many are required to get at least 1 significant digit correct in your answer?

Example 2 —Differentiation

Find $f'(3)$ for $f(x) = x^2$ using $f'(x) \approx \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

and $\Delta x = 0.2$

$$f'(3) = \frac{f(3+0.2) - f(3)}{0.2}$$

$$= \frac{f(3.2) - f(3)}{0.2} = \frac{3.2^2 - 3^2}{0.2} \;\; (\Delta x = 0.1) = \frac{10.24 - 9}{0.2} = \frac{1.24}{0.2} = 6.2$$

The actual value is

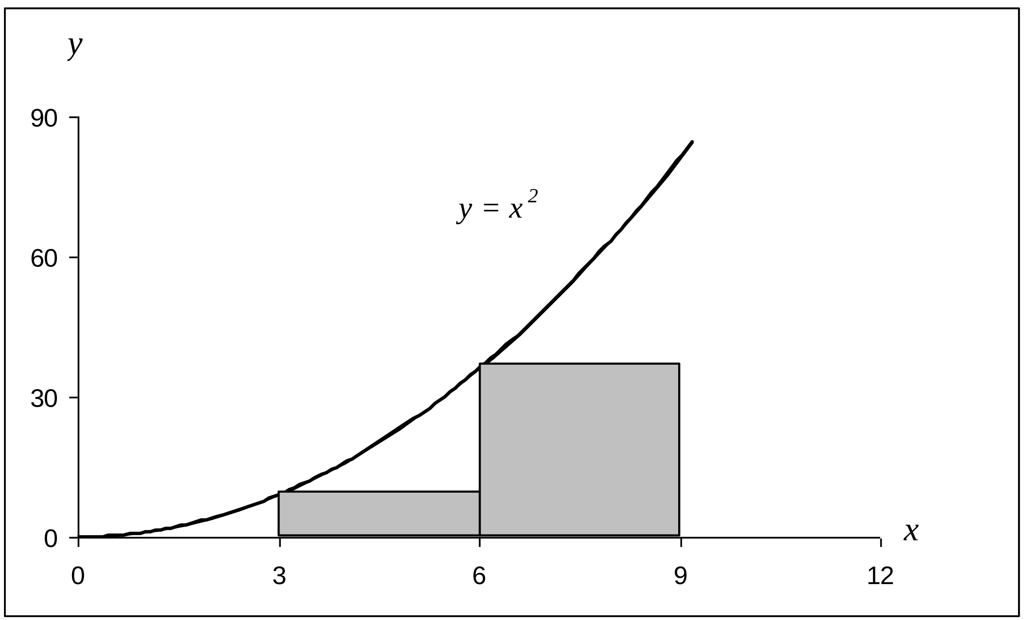
$$f'(x) = 2x, \qquad f'(3) = 2 \times 3 = 6$$

Truncation error is then, $6 - 6.2 = -0.2$

Can you find the truncation error with

Example 3 – Integration

Use two rectangles of equal width to approximate the area under the curve for $f(x) = x^2$ over the interval $[3,9]$



Integration example (cont.)

Choosing a width of 3, we have

$$\int_3^9 x^2 dx = (x^2)\Big|_{x=3}(6-3) + (x^2)\Big|_{x=6}(9-6)$$

$$= (3^2)3 + (6^2)3$$

$$= 27 + 108 = 135$$

Actual value is given by

$$\int_3^9 x^2 dx = \left[\frac{x^3}{3}\right]_3^9 = \left[\frac{9^3 - 3^3}{3}\right] = 234$$

Truncation error is then

$$234 - 135 = 99$$

Can you find the truncation error with 4 rectangles?

END of Presentation