Discrete Mathematics
CC318

Lecture 2

Propositional Logic - say a bit…

This week we're using propositional logic as a foundation for formal proofs.

Propositional logic is also the key to writing good code…you can't do any kind of conditional (if) statement without understanding the condition you're testing.

All the logical connectives we've discussed are also found in hardware and are called "gates."

Propositional Logic - 2 more defn…

A *tautology* is a proposition that's always TRUE.

A *contradiction* is a proposition that's always FALSE.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Propositional Logic - an infamous ≡

if NOT (blue AND NOT red) OR red then…

$$\neg(p \wedge \neg q) \vee q \equiv \neg p \vee q$$

$\neg(p \wedge \neg q) \vee q$	$\equiv (\neg p \vee \neg\neg q) \vee q$	DeMorgan's
	$\equiv (\neg p \vee q) \vee q$	Double negation
	$\equiv \neg p \vee (q \vee q)$	Associativity
	$\equiv \neg p \vee q$	Idempotent (unchanged)

Propositional Logic - one last proof

- *Show that $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.*
- *We use $\equiv$ to show that $[p \wedge (p \rightarrow q)] \rightarrow q \equiv T$.*

$[p \wedge (p \rightarrow q)] \rightarrow q$

⟶	$\equiv [p \wedge (\neg p \vee q)] \rightarrow q$	substitution for $\rightarrow$
⟶	$\equiv [(p \wedge \neg p) \vee (p \wedge q)] \rightarrow q$	distributive
⟶	$\equiv [\, F \vee (p \wedge q)] \rightarrow q$	uniqueness
⟶	$\equiv (p \wedge q) \rightarrow q$	identity
⟶	$\equiv \neg(p \wedge q) \vee q$	substitution for $\rightarrow$
⟶	$\equiv (\neg p \vee \neg q) \vee q$	DeMorgan's
⟶	$\equiv \neg p \vee (\neg q \vee q)$	associative
⟶	$\equiv \neg p \vee T$	excluded middle
⟶	$\equiv T$	domination

Predicate Logic - everybody loves somebody

Statement (Proposition), YES or NO?

- *3 + 2 = 5* YES
- *X + 2 = 5* NO
- *X + 2 = 5 for any choice of X in {1, 2, 3}* No
- *X + 2 = 5 for some X in {1, 2, 3}* YES

Predicate Logic - everybody loves somebody

Alicia eats pizza at least once a week.
Garrett eats pizza at least once a week.
Allison eats pizza at least once a week.
Gregg eats pizza at least once a week.
Ryan eats pizza at least once a week.
Meera eats pizza at least once a week.
Ariel eats pizza at least once a week.

⋮

Predicates

Alicia eats pizza at least once a week.

⋮

Define:

EP(x) = "x eats pizza at least once a week."

Universe of Discourse - x is a student in cs173

A predicate, or propositional function, is a function that takes some variable(s) as arguments and returns True or False.

Note that EP(x) is not a proposition, EP(Ariel) is.

Predicates

Suppose Q(x,y) = "x > y"

Statement(Proposition), YES or NO?

Q(x,y) NO
Q(3,4) YES
Q(x,9) NO

Predicate, YES or NO?

Q(x,y) YES
Q(3,4) NO
Q(x,9) YES

Predicates - the universal quantifier

Another way of changing a predicate into a proposition.

Suppose P(x) is a predicate on some universe of discourse.
Ex. B(x) = "x is carrying a backpack," x is set of cs173 students.

The universal quantifier of P(x) is the ***proposition****:*
"P(x) is true for all x in the universe of discourse."

We write it $\forall x\, P(x)$, and say "for all x, P(x)"

$\forall x\, P(x)$ is TRUE if P(x) is true for every single x.
$\forall x\, P(x)$ is FALSE if there is an x for which P(x) is false.

$\forall x\, B(x)$?

Predicates - the universal quantifier

B(x) = "x is wearing sneakers."
L(x) = "x is at least 21 years old."
Y(x)= "x is less than 24 years old."

Universe of discourse is people in this room.

Are either of these propositions true?

a) $\forall x\ (Y(x) \rightarrow B(x))$
b) $\forall x\ (Y(x) \vee L(x))$

A: only a is true

B: only b is true

C: both are true

D: neither is true

Predicates - the existential quantifier

Another way of changing a predicate into a proposition.

Suppose P(x) is a predicate on some universe of discourse.
Ex. C(x) = "x has a candy bar," x is set of cs173 students.

The existential quantifier of P(x) is the ***proposition****:*
"P(x) is true for some x in the universe of discourse."

We write it ∃x P(x), and say "for some x, P(x)"

∃x P(x) is TRUE if there is an x for which P(x) is true.
∃x P(x) is FALSE if P(x) is false for every single x.

∃x C(x)?

Predicates - the existential quantifier

B(x) = "x is wearing snea

L(x) = "x is at least 21 years old."

Y(x)= "x is less than 24 years old."

Universe of discourse is people in this room.

Are either of these propositions true?

a) $\exists x\, B(x)$

b) $\exists x\, (Y(x) \wedge L(x))$

A: only a is true

B: only b is true

C: both are true

D: neither is true

Predicates - more examples

L(x) = "x is a lion."
F(x) = "x is fierce.
C(x) = "x drinks coffee."

Universe of discourse is all creatures.

All lions are fierce.

$\forall x\ (L(x) \rightarrow F(x))$

Some lions don't drink coffee.

$\exists x\ (L(x) \wedge \neg C(x))$

Some fierce creatures don

$\exists x\ (F(x) \wedge \neg C(x))$

Predicates - more examples

B(x) = "x is a hummingbird."
L(x) = "x is a large bird."
H(x) = "x lives on honey."
R(x) = "x is richly colored."

Universe of discourse is all creatures.

All hummingbirds are richly colored.

$\forall x\ (B(x) \rightarrow R(x))$

No large birds live on honey.

$\neg\exists x\ (L(x) \wedge H(x))$

Birds that do not live on honey are dully colored.

$\forall x\ (\neg H(x) \rightarrow \neg R(x))$

Predicates - quantifier negation

Not all large birds live on hone[y] $\neg\forall x\,(L(x) \rightarrow H(x))$

$\forall x\ P(x)$ means "P(x) is true for every x."

What about $\neg\forall x\ P(x)$?

Not ["P(x) is true for every x."]

"There is an x for which P(x) is not true."

$\exists x\ \neg P(x)$

So, $\neg\forall x\ P(x)$ is the same as $\exists x\ \neg P(x)$.

$\exists x\ \neg(L(x) \rightarrow H(x))$

Predicates - quantifier negation

No large birds live on honey.

$\neg\exists x\ (L(x) \wedge H(x))$

$\exists x\ P(x)$ means "P(x) is true for some x."

What about $\neg\exists x\ P(x)$?

Not ["P(x) is true for some x."]

"P(x) is not true for all x."

$\forall x\ \neg P(x)$

So, $\neg\exists x\ P(x)$ is the same as $\forall x\ \neg P(x)$.

$\forall x\ \neg(L(x) \wedge H(x))$

Predicates - quantifier negation

So, $\neg\forall x\, P(x)$ is the same as $\exists x\, \neg P(x)$.
So, $\neg\exists x\, P(x)$ is the same as $\forall x\, \neg P(x)$.

General rule: to negate a quantifier, move negation to the right, changing quantifiers as you go.

Predicates - quantifier negation

No large birds live on honey.

$\neg\exists x\,(L(x) \wedge H(x)) \equiv \forall x\,\neg(L(x) \wedge H(x))$	Negation rule
$\equiv \forall x\,(\neg L(x) \vee \neg H(x))$	DeMorgan's
$\equiv \forall x\,(L(x) \rightarrow \neg H(x))$	Subst for $\rightarrow$

What's wrong with this proof?